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Noncommutative Mean Ergodic Theorem for Partial W^*-Dynamical Semigroups

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A noncommutative mean ergodic theorem for dynamical semigroups of maps on partial W^*-algebras of linear operators from a pre-Hilbert space into its completion is proved. This generalizes a similar result of Watanabe for dynamical semigroups of maps on W^*-algebras of operators.

1. INTRODUCTION

Noncommutative extensions of von Neumann's *mean ergodic theorem* for semigroups of linear contractions on Hilbert spaces have been established by a number of authors (Watanabe, 1979; Kovács and Szücs, 1966; Lance, 1976; Sinai and Anshelvich, 1976). The extensions are in the context of W^*-algebras and are useful in the study of the asymptotic averages of semigroups of bounded linear operators on Hilbert spaces. Studies of this type are encountered in the algebraic approach (Haag and Kastler, 1964; Bratelli and Robinson, 1979/1981) to quantum statistical mechanics or quantum field theory, where the fundamental object is a C^*-algebra or a W^*-algebra of observables.

The C^*-algebraic setting for quantum-theoretic studies is, however, restrictive, in view of the fact that observables are, in general, unbounded linear operators. Our main concern is with unbounded linear operators; to study collections of them algebraically, the C^*-algebraic setting must be abandoned. Accordingly, in this paper, the central structure is a partial $*$-algebra (Antoine *et al.*, 1990, 1991) of, in general, unbounded linear operators from a pre-Hilbert space to its completion. We consider semigroups of maps on such $*$-algebras and prove a *mean ergodic theorem* for the semigroups.

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The rest of the paper is organized as follows. In Section 2, we describe the algebraic setting in which we work. Partial W^*-dynamical semigroups on partial W^*-algebras are introduced in Section 3. The semigroups are generalizations of classical semigroups of maps on a W^*-algebra of operators. The mean ergodic theorem for partial W^*-dynamical semigroups is proved in Section 4. The result is a generalization of Watanabe's (1979) mean ergodic theorem for semigroups on W^*-algebras of operators.

2. THE ALGEBRAIC SETTING

A knowledge of partial *-algebras, as outlined in Antoine *et al.* (1990, 1991) is essential for what follows. Some other studies involving partial *-algebras and related to the present work have been described in Ekhaguere and Odiobala (1991) and Ekhaguere (1991a, b).

Let $\mathscr{A}$ be a partial *-algebra with partial multiplication $\bullet$ and involution *. For $x \in \mathscr{A}$, $L(x)$ [resp. $R(x)$] denotes the set of all *left* (resp. *right*) *multipliers* of x. If $\mathscr{C}$ is an arbitrary subset of $\mathscr{A}$, then

$L(\mathscr{C}) = \bigcap_{x \in \mathscr{C}} L(x)$, the set of *universal left multipliers* of $\mathscr{C}$

$R(\mathscr{C}) = \bigcap_{x \in \mathscr{C}} R(x)$, the set of *universal right multipliers* of $\mathscr{C}$

$M(\mathscr{C}) = L(\mathscr{C}) \cap R(\mathscr{C})$, the set of *universal multipliers* of $\mathscr{C}$

A member e of $\mathscr{A}$ is called a *unit* if $e \in M(\mathscr{C})$, $e^* = e$., and $e \bullet x = x = x \bullet e$, for all $x \in \mathscr{A}$. If $\mathscr{A}$ has a unit, then $\mathscr{A}$ is called *unital.*

A state on $\mathscr{A}$ is a sesquilinear form $\omega: \mathscr{A} \times \mathscr{A} \to \mathbb{C}$, the complex numbers, such that $\omega(e, e) = 1$.

We denote the set of all states on $\mathscr{A}$ by $S(\mathscr{A} \times \mathscr{A})$.

We shall employ certain concrete partial *-algebras called *partial O^*-algebras.* To describe these, let $\mathbb{D}$ be a pre-Hilbert space and $\mathbb{H}$ its Hilbert space completion, with inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$.

To the pair $(\mathbb{D}, \mathbb{H})$, we associate the set $L^+(\mathbb{D}, \mathbb{H})$ of all linear operators a with domain $D(a) = \mathbb{D}$, range of a contained in $\mathbb{H}$, and $D(a^*) \supset \mathbb{D}$, where x^* is the operator adjoint of x. Then, $L^+(\mathbb{D}, \mathbb{H})$ is a linear space when supplied with the usual notions of addition and scalar multiplication. This linear space becomes a partial *-algebra with involution $^+$, and partial multiplication $\Box$ defined as follows.

For $a \in L^+(\mathbb{D}, \mathbb{H})$,

$$a^+ = a^* \lceil \mathbb{D}$$

and for a, $b \in L^+(\mathbb{D}, \mathbb{H})$ such that $a^+\mathbb{D} \subseteq D(b^*)$ and $b\mathbb{D} \subseteq D(a^{+*})$,

$$a \Box b = a^{+*} b$$

We shall denote the partial *-algebra $(L^+(\mathbb{D}, \mathbb{H}), {}^+, \Box)$ simply by $L_w^+(\mathbb{D}, \mathbb{H})$. This partial *-algebra is unital; we denote its unit by e.

Definition. A partial *-subalgebra of $L_w^+(\mathbb{D}, \mathbb{H})$ is called a *partial O^*-algebra.*

Remark. 1. It follows that $L_w^+(\mathbb{D}, \mathbb{H})$ is the maximal partial O^*-algebra corresponding to the pair $(\mathbb{D}, \mathbb{H})$.

2. Let $\mathcal{M} \subset L_w^+(\mathbb{D}, \mathbb{H})$ be a partial O^*-algebra. In addition to the weak topology t_w, σ-weak topology $t_{\sigma w}$, the strong topology t_s, and σ-strong* topology $t_{\sigma s^*}$ on $\mathcal{M}$ explained in Antoine *et al.* (1990, 1991) and Ekhaguere (1991a), we shall also employ the t_{λ^*}-topology. This arises as follows.

Let $\mathcal{M}_+$ be the positive portion of $\mathcal{M}$. Denote $\mathcal{M}_+ \cup \{e\}$ by $\mathcal{M}_e$. For $a \in \mathcal{M}_e$, let

$$\lambda_a^*(x) = \sup_{\xi \in \mathbb{D}} \mathbb{D}\left(\frac{\|x\xi\|_{\mathbb{H}} + \|x^+\xi\|_{\mathbb{H}}}{\|a\xi\|_{\mathbb{H}}}\right), \qquad x \in \mathcal{M}$$

where $\alpha/0 = \infty$ for $\alpha > 0$; and set

$$\mathcal{M}^a = \{x \in \mathcal{M} : \lambda_a^*(x) < \infty\}$$

Then, for each $a \in \mathcal{M}$, $\lambda_a^*(\cdot)$ is a norm on $\mathcal{M}^a$. Furthermore, if $a, b \in \mathcal{M}_e$, then $\mathcal{M}^a + \mathcal{M}^b \subset \mathcal{M}^{a^+ \Box a + b^+ \Box b + a + b + e}$, showing that the family $\{(\mathcal{M}^a, \lambda_a^*(\cdot)) : a \in \mathcal{M}_e\}$ is directed and covers $\mathcal{M}$. For $a \in \mathcal{M}_e$, let j_a be the injection of $\mathcal{M}^a$ in $\mathcal{M}$. Then the topology t_{λ^*} is the inductive topology on $\mathcal{M}$ determined by the collection $\{(\mathcal{M}^a, j_a) : a \in \mathcal{M}_e\}$. The t_{λ^*}-topology reduces to the uniform topology on the space $B(\mathbb{H})$ of continuous linear maps on $\mathbb{H}$. Arnal and Jurzak (1977) discussed an analogous topology called the λ-topology.

Notation. If $\mathcal{N} \subseteq \mathcal{M}$, define the commutants $\mathcal{N}'_\sigma$ and $\mathcal{N}'_\Box$ by

$$\mathcal{N}'_\sigma = \{y \in L^+(\mathbb{D}, \mathbb{H}) : \langle x\xi, y\eta \rangle_{\mathbb{H}} = \langle y^+\xi, x^+\eta \rangle_{\mathbb{H}} \text{ for all } \xi, \eta \in \mathbb{D} \text{ and } x \in \mathcal{N}\}$$

$$\mathcal{N}'_\Box = \{y \in N'_\sigma : y \in L(x) \cap R(x) \text{ and } y \Box x = x \Box y \text{ for all } x \in \mathcal{N}\}$$

These are unbounded commutants of the set $\mathcal{N}$. (Antoine *et al.*, 1990, 1991).

3. PARTIAL W^*-DYNAMICAL SEMIGROUPS

A unital partial O^*-algebra $\mathcal{M} \subset L_w^+(\mathbb{D}, \mathbb{H})$ is called a *partial W^*-algebra* (Ekhaguere, 1991a) if $\mathcal{M}$ is self-adjoint [i.e., $\bigcap_{a \in \mathcal{M}} D(a^*) = \mathbb{D}$], $t_{\sigma w}$-closed, and $R(\mathcal{M})$ is $t_{\sigma s^*}$-dense in $\mathcal{M}$.

In the sequel, *$\mathcal{M}$ is a partial W^*-algebra*, $\mathcal{M}^1 = \mathcal{M}$, and for $n \geqslant 2$, $\mathcal{M}^n$ is the n-fold Cartesian product of $\mathcal{M}$ with itself. If τ is a topology on $\mathcal{M}$, write τ^n, $n \geqslant 2$, for the product topology $\tau \times \tau \times \cdots \times \tau$ (n times).

Introduce the following spaces:

$$\mathcal{M}^n_{*\sigma w} = \text{the linear space of all } (t_{\sigma w})^n\text{-continuous linear forms on } \mathcal{M}$$

$$\mathcal{M}^n_{*\lambda} = \text{the linear space of all } (t_{\lambda *})^n\text{-continuous linear forms on } \mathcal{M}$$

$$\mathcal{M}^n_{*} = \mathcal{M}^n_{*\sigma w} \cap \mathcal{M}^n_{*\lambda}$$

On the space $\mathcal{M}^n_*$ we shall consider the weak topology $\sigma(\mathcal{M}^n_*, \mathcal{M}^n)$ as well as the topology t_{*u} of uniform convergence on the $t_{\lambda *}$-bounded subsets of $\mathcal{M}$. We remark that the dual of $(\mathcal{M}^n_*, \sigma(\mathcal{M}^n_*, \mathcal{M}^n))$ is precisely $\mathcal{M}^n$.

Definition. A map $\Phi: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is called

1. *conjugate-bilinear* if

   (a) $\Phi(x, y)^+ = \Phi(y, x)$

   (b) $\Phi(x, \alpha y + \beta z) = \alpha\Phi(x, y) + \beta\Phi(x, z), \forall x, y, z \in \mathcal{M}, \alpha, \beta \in \mathbb{C}$

2. *completely positive* if

$$\sum_{i,j=1}^{n} \langle \xi_i, \Phi(x_i, x_j)\, \xi_j \rangle_{\mathbb{H}} \geqslant 0$$

   for each positive integer n and all $\xi_k \in \mathbb{D}$, $x_k \in \mathcal{M}$, $k = 1, 2, \ldots, n$.

Remark. The class of completely positive conjugate bilinear maps on arbitrary partial *-algebras was studied in Ekhaguere and Odiobala (1991).

Notation. 1. Write $CP(\mathcal{M} \times \mathcal{M})$ for the set of all completely positive conjugate-bilinear maps (Ekhaguere and Odiobala, 1991) from $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$, and $CP(\mathcal{M} \times \mathcal{M})_{\sigma w}$ [resp. $S(\mathcal{M} \times \mathcal{M})_{\sigma w}$] for the subset of $CP(\mathcal{M} \times \mathcal{M})$ [resp. $S(\mathcal{M} \times \mathcal{M})$] consisting of those members of $CP(\mathcal{M} \times \mathcal{M})$ [(resp. $S(\mathcal{M} \times \mathcal{M})$] that are continuous from $(\mathcal{M} \times \mathcal{M}, t_{\sigma w} \times t_{\sigma w})$ into $(\mathcal{M}, t_{\sigma w})$.

2. If $\Phi_1, \Phi_2 \in CP(\mathcal{M} \times \mathcal{M})$, then define $\Phi_1 \circ \Phi_2$ by $(\Phi_1 \circ \Phi_2)(x, y) = \Phi_1(e, \Phi_2(x, y))$, $x, y \in \mathcal{M}$.

Similarly, if $\omega \in S(\mathcal{M} \times \mathcal{M})$ and $\Phi \in CP(\mathcal{M} \times \mathcal{M})$, define $\omega \circ \Phi$ by $(\omega \circ \Phi)(x, y) = \omega(e, \Phi(x, y))$, $x, y \in \mathcal{M}$.

3. We define the map $\imath: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by

$$\imath(x, y) = t_{\sigma w} \times t_{\sigma w} - \lim_{\alpha, \beta} (x_\alpha^+ \,\square\, y_\beta)$$

for arbitrary nets (x_α), (y_β) in $R(\mathcal{M})$ such that

$$x_\alpha \xrightarrow{t_{\sigma s^*}} x \text{ and } y_\beta \xrightarrow{t_{\sigma s^*}} y$$

We remark that $\imath$ is *idempotent* (i.e. $\imath \circ \imath = \imath$), $\imath(e, e) = e$, and $\imath \in CP(\mathcal{M} \times \mathcal{M})_{\sigma w}$.

Remark. The following notion was introduced in Ekhaguere (1991a).

Definition. A *partial* W^**-dynamical semigroup* (p.d.s.) on $\mathcal{M}$ is a one-parameter family $\varphi_{[0,\infty)} = \{\varphi_t : t \geqslant 0\}$ of members of $CP(\mathcal{M} \times \mathcal{M})_{\sigma w}$ such that:

(i) $\varphi_0 = \imath$.

(ii) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for arbitrary $s, t \geqslant 0$.

(iii) For fixed $(x, y) \in \mathcal{M} \times \mathcal{M}$, the map $t \to \varphi_t(x, y)$ is continuous from $[0, \infty)$ to $(\mathcal{M}, t_{\sigma w})$.

Remark. 1. A description of a class of p.d.s. is furnished in Ekhaguere (1991a, Theorem 3.9).

2. The class of p.d.s. discussed in the sequel is introduced as follows.

Definition. A p.d.s. $\varphi_{[0,\infty)} = \{\varphi_t : t \geqslant 0\}$ on $\mathcal{M}$ will be called

(i) t_{λ^*}*-equicontinuous* if for each $a \in \mathcal{M}_e$ there are $b(a), c(a) \in \mathcal{M}_e$ such that

$$\lambda_a(\varphi_t(x, y)) \leqslant \lambda_{b(a)}(x)\, \lambda_{c(a)}(y), \qquad \forall x, y \in \mathcal{M}, \quad t \geqslant 0$$

(ii) a p.d.s. *with squares* if $\varphi(e, x)^+$ lies in $L(\varphi(e, x))$ for all $x \in \mathcal{M}$ and $t \geqslant 0$.

Remark. 1. Any p.d.s. $\varphi_{[0,\infty)} = \{\varphi_t : t \geqslant 0\}$ with values in $M(\mathcal{M})$ is a p.d.s. with squares.

2. A p.d.s. $\varphi_{[0,\infty)} = \{\varphi_t : t \geqslant 0\}$ with squares satisfies the generalized Cauchy–Schwarz inequality established in Ekhaguere (1991a).

3. A member ω of $S(\mathcal{M}\times\mathcal{M})$ is called $\varphi_{[0,\infty)}$-*invariant* if $\omega\circ\varphi_t=\omega$ on $\mathcal{M}\times\mathcal{M}$.

4. If $\varphi_{[0,\infty)}=\{\varphi_t\colon t\geqslant 0\}$ is a $t_{\lambda*}$-equicontinuous p.d.s. and x, y lie in $t_{\lambda*}$-bounded subsets of $\mathcal{M}$, then $\varphi_{[0,\infty)}=\{\varphi_t\colon t\geqslant 0\}$ is a uniformly $t_{\lambda*}$-bounded subset of $\mathcal{M}$.

5. Let $\varphi_{[0,\infty)}=\{\varphi_t\colon t\geqslant 0\}$ be a $t_{\lambda*}$-equicontinuous p.d.s. and $\omega\in\mathcal{M}_*$. If $f\in L^1(\mathbb{R}, dt)$ and (x, y) lies in a $(t_{\lambda*})^2$-bounded subset $\mathscr{B}$ of $\mathcal{M}\times\mathcal{M}$, then the integral $\int_0^\infty dt\,\omega(\varphi_t(x, y))\,f(t)$ exists. Hence, $\omega\mapsto\int_0^\infty dt\,\omega(\varphi_t(x, y))\,f(t)$ is a linear form on $(\mathcal{M}'_*, \sigma(\mathcal{M}_*, \mathcal{M}))$ for arbitrary $(x, y)\in\mathscr{B}$, showing that there is a member $\varphi(x, y; f)$ of $\mathcal{M}$ such that

$$\int_0^\infty dt\,\omega(\varphi_t(x, y))\,f(t)=\omega(\varphi(x, y; f))$$

for arbitrary $(x, y)\in\mathscr{B}$. We denote $\varphi(x, y; f)$ by $\int_0^\infty dt\,\varphi_t(x, y)\,f(t)$, $f\in L^1(\mathbb{R}, dt)$, $(x, y)\in\mathscr{B}$. If $T\in(0, \infty)$, $\chi_{[0,T)}$ is the indicator function of the set $[0, T)$, and $f=\chi_{[0,T)}$, we write the last integral simply as $\int_0^T dt\,\varphi_t(x, y)$, $(x, y)\in\mathscr{B}$. In the next section, an *ergodic theorem* concerning this integral is established.

4. A MEAN ERGODIC THEOREM

We prove the following ergodic theorem for partial W^*-dynamical semigroups.

Mean Ergodic Theorem. Let $\mathbb{H}$ be a Hilbert space, $\mathbb{D}$ a dense subspace of $\mathbb{H}$, $\mathcal{M}\subset L_w^+(\mathbb{D}, \mathbb{H})$ a partial W^*-algebra, and $\varphi_{[0,\infty)}=\{\varphi_t\colon t\geqslant 0\}$ a partial W^*-dynamical semigroup on $\mathcal{M}$ with squares. Let $\omega\in S(\mathcal{M}\times\mathcal{M})_{\sigma w}$ be a nondegenerate, GNS-representable, $\varphi_{[0,\infty)}$-invariant state on $\mathcal{M}$. Then:

1. For any (x, y) lying in a $(t_{\lambda*})^2$-bounded subset of $\mathcal{M}$, the limit

$$\varepsilon(x, y)\equiv t_{\sigma w}-\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)$$

exists.

2. ε is idempotent, nondegenerate, and satisfies

$$\omega\circ\varepsilon=\omega$$

$$\varepsilon\circ\varphi_t=\varphi_t\circ\varepsilon=\varepsilon, \qquad \forall t\geqslant 0$$

on $(t_{\lambda*})^2$-bounded subsets of $\mathcal{M}\times\mathcal{M}$.

Proof. Since ω is GNS-representable (Antoine *et al.*, 1990, 1991), it gives rise to a GNS triple $(\pi_\omega, \mathbb{H}_\omega, \xi_\omega)$ consisting of a cyclic representation π_ω, with cyclic vector ξ_ω, of $\mathscr{M}$ in $L_\omega^+(\mathbb{D}_\omega, \mathbb{H}_\omega)$, where $\mathbb{H}_\omega$ is a Hilbert space and $\mathbb{D}_\omega$ is some dense subspace of $\mathbb{H}_\omega$. Let $(\mathscr{M}\times\mathscr{M})^+ = \{(x, y)\in\mathscr{M}\times\mathscr{M}: \imath(x, y)^+ = \imath(x, y)\}$ and $\mathbb{K}_\omega =$ the norm closure in $\mathbb{H}_\omega$ of $\{\pi_\omega(\imath(x, y))\,\xi_\omega: (x, y)\in(\mathscr{M}\times\mathscr{M})^+\}$. Then $\mathbb{K}_\omega$ is a real Hilbert subspace of $\mathbb{H}_\omega$. Define P_t on the dense subspace $\pi_\omega(\imath((\mathscr{M}\times\mathscr{M})^+))\,\xi_\omega$ of $\mathbb{K}_\omega$ by

$$P_t\pi_\omega(\imath(x, y))\,\xi_\omega = \pi_\omega(\varphi_t(x, y))\,\xi_\omega, \qquad (x, y)\in(\mathscr{M}\times\mathscr{M})^+$$

Since $(e, \varphi_t(x, y))\in(\mathscr{M}\times\mathscr{M})^+$ whenever $(x, y)\in(\mathscr{M}\times\mathscr{M})^+$ and

$$P_s\pi_\omega(\imath(x, y))\,\xi_\omega = \pi_\omega(\varphi_s(x, y))\,\xi_\omega = \pi_\omega(\imath(e, \varphi_s(x, y)))\,\xi_\omega$$

it follows that

$$\begin{aligned} P_tP_s\pi_\omega(\imath(x, y))\,\xi_\omega &= P_t\pi_\omega(\imath(e, \varphi_s(x, y)))\,\xi_\omega \\ &= \pi_\omega(\varphi_t(e, \varphi_s(x, y)))\,\xi_\omega \\ &= \pi_\omega(\varphi_{t+s}(x, y))\,\xi_\omega \\ &= P_{t+s}\pi_\omega(\imath(x, y))\,\xi_\omega \end{aligned}$$

for all $(x, y)\in(\mathscr{M}\times\mathscr{M})^+$, $t, s\geqslant 0$, showing that $\{P_t: t\geqslant 0\}$ is a semigroup. Moreover, each P_t is a contraction, since for arbitrary $(x, y)\in(\mathscr{M}\times\mathscr{M})^+$ and $t\geqslant 0$,

$$\begin{aligned} &\|P_t\pi_\omega(\imath(x, y))\,\xi_\omega\|^2_{\mathbb{K}_\omega} \\ &\quad= \langle\pi_\omega(\varphi_t(x, y))\,\xi_\omega, \pi_\omega(\varphi_t(x, y))\,\xi_\omega\rangle_{\mathbb{K}_\omega} \\ &\quad= \langle\xi_\omega, \pi_\omega(\varphi_t(e, \imath(x, y))\,\Box\,\pi_\omega(\varphi_t(e, \imath(x, y))\,\xi_\omega\rangle_{\mathbb{K}_\omega} \\ &\qquad\qquad\text{[since } \omega \text{ is GNS-representable and } \{\varphi_t: t\geqslant 0\} \\ &\qquad\qquad\text{is a semigroup with squares]} \\ &\quad\leqslant\langle\xi_\omega, \pi_\omega(\varphi_t(\imath(x, y), \imath(x, y)))\,\xi_\omega\rangle_{\mathbb{K}_\omega} \\ &\quad= \omega(e, \varphi_t(\imath(x, y), \imath(x, y))) \\ &\quad= \omega(\imath(x, y), \imath(x, y)) \\ &\qquad\qquad\text{[since } \omega \text{ is } \varphi_{[0,\infty)}\text{-invariant]} \\ &\quad= \|\pi_\omega(\imath(x, y))\,\xi_\omega\|^2_{\mathbb{K}_\omega} \end{aligned}$$

showing that P_t is a contraction on $\pi_\omega(\imath((\mathscr{M}\times\mathscr{M})^+))\,\xi_\omega$. We denote the extension of P_t to the whole of $\mathbb{K}_\omega$ again by P_t.

Let E be the orthogonal projection of $\mathbb{K}_\omega$ onto the linear span of $\{\xi\in\mathbb{K}_\omega: P_t\xi = \xi \text{ for all } t\geqslant 0\}$. By the Mean Ergodic Theorem (von

Neumann, 1932; Birkhoff, 1939; Dunford and Schwartz, 1957) for contraction semigroups, the net $\{(1/T)\int_0^T dt\, P_t\}_{T\in(0,\infty)}$ converges strongly to E. Hence, for arbitrary (x, y) in a $(t_{\lambda*})^2$-bounded subset $\mathscr{B}^+$ of $(\mathscr{M}\times\mathscr{M})^+$ and $a\in\pi_\omega(\mathscr{M})'_\square$, we have

$$\begin{aligned}\left\langle\left(\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)\square a\right)\xi, \eta\right\rangle_{\mathbb{K}_\omega} &= \left\langle\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)^{+*} a\xi, \eta\right\rangle_{\mathbb{K}_\omega}\\ &= \left\langle a\xi, \pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)^{+}\eta\right\rangle_{\mathbb{K}_\omega}\\ &= \left\langle\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)\xi, a^+\eta\right\rangle_{\mathbb{K}_\omega}\end{aligned}$$

whence

$$\lim_{T\to\infty}\left\langle\left(\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)\square a\right)\xi, \eta\right\rangle_{\mathbb{K}_\omega} = \langle E\pi_\omega(\imath(x, y))\,\xi, a^+\eta\rangle_{\mathbb{K}_\omega}$$

showing that $E\pi_\omega(\imath(x, y))\colon \mathscr{M}\to D(a^{+*})$. Hence

$$\lim_{T\to\infty}\left\langle\left(\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)\square a\right)\xi, \eta\right\rangle_{\mathbb{K}_\omega} = \langle a\square(E\pi_\omega(\imath(x, y)))\,\xi, \eta\rangle_{\mathbb{K}_\omega}$$

i.e.,

$$t_w\text{-}\lim_{T\to\infty}\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right)\square a = a\square(E\pi_\omega(\imath(x, y)))$$

for all $(x, y)\in\mathscr{B}^+$ and $a\in\pi_\omega(\mathscr{M})'_\square$, whence

$$t_w\text{-}\lim_{T\to\infty}\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right) = E\pi_\omega(\imath(x, y))$$

It follows that the net $\{(1/T)\int_0^T dt\,\varphi_t(x, y)\}_{T\in(0,\infty)}$ converges weakly for all (x, y) lying in any $(t_{\lambda*})^2$-bounded subset of $\mathscr{M}\times\mathscr{M}$. As the net $\{(1/T)\int_0^T dt\,\varphi_t(x, y)\}_{T\in(0,\infty)}$ is contained in a uniformly $t_{\lambda*}$-bounded subset of $\mathscr{M}$ whenever (x, y) lies in a $(t_{\lambda*})^2$-bounded subset of $\mathscr{M}\times\mathscr{M}$, the net $\{\pi_\omega((1/T)\int_0^T dt\,\varphi_t(x, y))\}_{T\in(0,\infty)}$ is also contained in a $t_{\lambda*}$-bounded of $\pi_\omega(\mathscr{M})$. But on $t_{\lambda*}$-bounded subsets, the weak topology t_w coincides (Ekhaguere, 1991a) with the σ-weak topology $t_{\sigma w}$. Hence

$$t_{\sigma w}\text{-}\lim_{T\to\infty}\pi_\omega\left(\frac{1}{T}\int_0^T dt\,\varphi_t(x, y)\right) = E\pi_\omega(\imath(x, y))$$

As $\pi_\omega(\mathscr{M})$ is $t_{\sigma w}$-closed, there is a unique member $\varepsilon(x, y)$ of $\mathscr{M}$ such that

$$t_{\sigma w}\text{-}\lim_{T\to\infty} \pi_\omega\left(\frac{1}{T}\int_0^T dt\, \varphi_t(x, y)\right) = \pi_\omega((x, y))$$

and, as π_ω is faithful,

$$t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \varphi_t(x, y) = \varepsilon(x, y)$$

for all (x, y) lying in a $(t_{\lambda *})^2$-bounded subset of $\mathscr{M}\times\mathscr{M}$

2. The map ε has the alleged properties. Indeed, from the definition of ε, we have

$$\begin{aligned}(\omega\circ\varepsilon)(x, y) &= \omega(e, \varepsilon(x, y))\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \omega(e, \varphi_t(x, y))\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \omega(x, y)\\ &\qquad [\text{since } \omega \text{ is } \varphi_{[,0\infty)}\text{-invariant}]\\ &= \omega(x, y), \qquad \text{for arbitary } (x, y) \text{ lying in a}\\ &\qquad (t_{\lambda *})^2\text{-bounded subset of } \mathscr{M}\times\mathscr{M}\end{aligned}$$

and also

$$\begin{aligned}(\varepsilon\circ\varphi_t(x, y) &= \varepsilon(e, \varphi_t(x, y))\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T ds\, \varphi_s(e, \varphi(x, y))\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T ds\, \varphi_{t+s}(x, y)\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^T ds\, \varphi_t(e, \varphi_s(x, y))\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \varphi_t\left(e, \frac{1}{T}\int_0^T ds\, \varphi_s(x, y)\right)\\ &= \varphi_t(e, \varepsilon(x, y))\\ &= t_{\sigma w}\text{-}\lim_{T\to\infty} \frac{1}{T}\int_0^{t+T} d\tau\, \varphi_\tau(x, y)\\ &= \varepsilon(x, y)\end{aligned}$$

for arbitrary (x, y) lying in a $(t_{\lambda^*})^2$-bounded subset of $\mathscr{M} \times \mathscr{M}$. This concludes the proof.

Remark. The foregoing theorem generalizes Watanabe's (1979) noncommutative mean ergodic theorem obtained for semigroups on W^*-algebras of bounded operators.

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